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Water Waves in a Nonhomogeneous Incompressible Fluid

by

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Water Waves in a Nonhomogeneous

Incompressible Fluid

by

A. E. Green[†] and P. M. Naghdi[‡]

Abstract. After a brief discussion of some undesirable features of a number of different partial differential equations often employed in the existing literature on water waves, a relatively simple restricted theory is constructed by a direct approach which is particularly suited for applications to problems of fluid sheets. The rest of the paper is concerned with a derivation of a system of nonlinear differential equations (which may include the effects of gravity and surface tension) governing the two-dimensional motion of incompressible inviscid fluids for propagation of fairly long waves in a nonhomogeneous stream of water of variable initial depth, as well as some new results pertaining to hydraulic jumps. The latter includes an additional class of possible solutions not noted previously.

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1. Introduction

Frequently it is difficult to find exact solutions of systems of linear or nonlinear differential equations which characterize either a specific problem or a class of problems in mechanics. For this reason, in many areas of mechanics of solids and fluids, widespread use has been made in recent years of approximate perturbation methods, especially when the system of differential equations contains one or more small parameters. Usually it is not possible to prove rigorously that the approximate solutions are related to exact solutions of the equations in a precise way. Nevertheless, considerable confidence is often placed in the results, partly because they may be shown to be satisfactory in special cases for which exact results are available. A slightly different line of thought is to replace a system of nonlinear differential equations either by an approximate linear system of the same order, or by a nonlinear system of lower order. In the former case the linearization method is clearcut and leads to a unique system of *equations*. In the latter, the situation is often very confused: the procedure is singular in that it reduces the order of the equations and may also lead to some related dilemmas or questions concerning the associated boundary and initial conditions. Moreover, the procedure is not unique since it may lead to sets of apparently equally justifiable equations whose characters are quite different. This is particularly true in the derivation of nonlinear differential equations which are employed in the study of the propagation of fairly long water waves, as can be seen from accounts of the subject given by Peregrine [1] and by Whitham [2]. In fact, according to Whitham [2, p. 463], "The...derivations allow great flexibility and the approach naturally allows the various alternatives."

We list below from [2] some of the alternative forms of equations for water waves moving in the direction of a fixed x-axis for a stream of initial constant depth h. Let the elevation of the stream be $h + \eta$ and let u denote the horizontal velocity, where u, η are functions of x and the time t. Further, recall from [2, pp. 460-463] the system of equations

$$\begin{aligned} &\eta_t + \{u(h+\eta)\}_x = 0 \quad , \\ &u_t + uu_y + g\eta_x + \frac{1}{3} c^2 h \eta_{xxx} = 0 \end{aligned} \tag{1}$$

and the pair of equations attributed to Boussinesq, i.e.,

$$\begin{aligned} &\eta_t + \{u(h+\eta)\}_x = 0 \quad , \\ &u_t + uu_x + g\eta_x + \frac{1}{3} h \eta_{xtt} = 0 \quad , \end{aligned} \tag{2}$$

where subscripts indicate partial differentiation with respect to t or x, $c^2 = gh$ and g is the acceleration due to gravity. Both systems of equations (1) and (2) allow for wave propagation in either direction along the x-axis. For waves moving along the positive x-direction only there is the Korteweg-deVries (hereafter referred to as the K.dV.) equation [3] given by

$$\eta_t + c(1 + \frac{3}{2}\frac{\eta}{h})\eta_x + \frac{1}{6} ch^2 \eta_{xxx} = 0 \tag{3}$$

or the following equation due to Benjamin et al. [4]:

$$\eta_t + c(1 + \frac{3}{2}\frac{\eta}{h})\eta_x - \frac{1}{6} ch^2 \eta_{xxt} = 0 \quad . \tag{4}$$

Some of the properties of the differential equations (1) to (4) should be noted. It may immediately be verified that the set of equations (2) and (4) have <u>steady state</u> solutions only if η and u are both constants. Also, although the K.dV. equation (3) admits a solitary wave in which the velocity at infinity is zero when the stream there is at its undisturbed height h, it does <u>not</u> admit a steady state solution with u constant and $\eta = 0$ at infinity. This fact is related to another property of (3) which is also shared by (2) and (4): the three sets of equations (2) to (4) are not invariant in form under a <u>constant</u>

superposed rigid body motion of the whole fluid. To see this, suppose that a constant superposed rigid body translational velocity k is imposed on the whole fluid so that the particles at the place x are displaced to x^+ at time t^+ where

$$x^+ = x + kt \quad , \quad t^+ = t + \bar{k} \quad , \quad (\bar{k} \text{ constant}) \quad , \tag{5}$$

and the velocity u is replaced by $u + k$. Then, if the dependent variables x,t in (2) to (4) are changed to (5), the equations for u, η in terms of x^+, t^+ are different from those in terms of x,t. For example, equation (3) becomes

$$\eta_{t^+} + k\eta_{x^+} + c(1 + \frac{3}{2}\frac{\eta}{h})\eta_{x^+} + \frac{1}{6} ch^2 \eta_{x^+x^+x^+} = 0 \quad . \tag{6}$$

This means that the character of the solutions of (2), (3) and (4) are radically altered by superposing a <u>constant</u> rigid body translational velocity on the fluid, which is contrary to what happens if we use the full three-dimensional equations of motion for an inviscid fluid. On the other hand, the set of equations (1) are not subject to this drawback, and they do have useful steady state solutions. It may be argued that because of the nature of the approximation in obtaining (2) to (4) from the three-dimensional theory we should not expect these equations to be invariant under a superposed constant translational velocity, but this then leaves in doubt which version of any of the sets (2) to (4) are correct and are to be chosen as basic. The difficulty disappears if we linearize any of the above sets since the resulting equations are then invariant under a <u>small</u> superposed constant translational velocity, as we would expect.

It might appear from the above discussion that the equations (1) may be preferable to any of (2) to (4), but arguments are put forward in [2, p. 462] to suggest that the set (2) is to be preferred to (1). Although considerable use has been made of some of the equations (1) to (4), it would seem that they all rest on a somewhat shaky physical foundation.

In recent years the derivation of nonlinear equations which are suitable for the propagation of fairly long water waves has been approached from a completely different point of view, namely via the theory of a directed (or Cosserat) surface. A direct two-dimensional theory of this kind is established in [5,6] for water waves employing integral balance laws for mass conservation, momentum, moment of momentum and (when required) energy, in parallel with what is done in the development of the three-dimensional theory, and is valid for water of variable initial depth. Such equations are fully nonlinear and satisfy all usual invariance requirements as do their three-dimensional counterparts. The resulting field equations of this theory contain certain assigned fields and inertia coefficients which are unspecified, but these are identified by an appeal to rather easily accessible results which can be deduced from the three-dimensional equations. An alternative derivation given in [7] starts from the three-dimensional energy equation and, after employing a single approximation for the velocity field, proceeds only with the help of invariance conditions under superposed rigid body motions and leads to the same system of equations as those in [5,6].

The present paper is concerned with some extensions of previous work [5,6] by a direct approach. First, we briefly construct a restricted theory of directed surfaces in which some of the kinematic and kinetic variables are restricted at the outset. This restricted theory, which may be regarded as a special case of a more general theory of a Cosserat surface, is particularly useful in application to problems of fluid sheets and is utilized in the rest of the paper for incompressible fluids in which the mass density may vary with depth. A nonlinear system of (two-dimensional) differential equations is derived for propagation of water waves in a nonhomogeneous stream of variable initial depth [see Eqs. (35), (54) and (55)] and some extensions of earlier work [6, section 8] on hydraulic jumps are discussed.

2. General Background. A Restricted Theory of Directed Surfaces.

We first discuss, in this section, some background information from the theory of a Cosserat (or directed) surface and then summarize a special case of the theory in a manner which is particularly suitable for application to the propagation of water waves in a nonhomogeneous fluid. We recall that a Cosserat surface $\mathcal{C}$ comprises a material surface (embedded in a Euclidean 3-space) and a single deformable director attached to every material point (or particle) of $\mathcal{C}$. Let the particles of the material surface of $\mathcal{C}$ be identified with a system of convected coordinates† θ^α $(\alpha = 1,2)$ and let the surface occupied by the material surface of $\mathcal{C}$ in the present configuration at time t be referred to by $\mathcal{s}$. Let $\mathbf{r}$ and $\mathbf{d}$ denote the position vector of a typical point of $\mathcal{s}$ and the director at the same point, respectively, and also designate the base vectors along the θ^α-curves on $\mathcal{s}$ by $\mathbf{a}_\alpha$. Then, a motion of the Cosserat surface is defined by vector-valued functions which assign position $\mathbf{r}$ and director $\mathbf{d}$ to each particle of $\mathcal{C}$ at each instant of time, i.e.,‡

$$\mathbf{r} = \mathbf{r}(\theta^\alpha, t) \quad , \quad \mathbf{d} = \mathbf{d}(\theta^\alpha, t) \quad , \quad [\mathbf{a}_1 \mathbf{a}_2 \mathbf{d}] > 0 \tag{7}$$

and the condition $(7)_3$ ensures that the director $\mathbf{d}$ is nowhere tangent to $\mathcal{s}$. The base vectors $\mathbf{a}_\alpha$ and their reciprocals $\mathbf{a}^\alpha$, the unit normal $\mathbf{a}_3$ and the components of the metric tensors $a_{\alpha\beta}$ and $a^{\alpha\beta}$ at each point of $\mathcal{s}$ are defined by

$$\begin{gathered}\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \quad , \quad \mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta \quad , \quad a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad , \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta \quad , \\ a^{\frac{1}{2}} \mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 \quad , \quad a = \det a_{\alpha\beta} \quad , \quad a^{\frac{1}{2}} = [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3] > 0 \quad ,\end{gathered} \tag{8}$$

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†Recall that when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates associated with each material point remain the same for all time.

‡The choice of positive sign in $(7)_3$ is for definiteness. Alternatively, it will suffice to assume that $[\mathbf{a}_1 \mathbf{a}_2 \mathbf{d}] \neq 0$ with the understanding that in any given motion the scalar triple product $[\mathbf{a}_1 \mathbf{a}_2 \mathbf{d}]$ is either >0 or <0.

where δ^{α}_{β} is the Kronecker delta. The velocity and the director velocity vectors are given by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w} = \dot{\underset{\sim}{d}} \quad , \tag{9}$$

where a superposed dot denotes differentiation with respect to t holding θ^{α} fixed. Throughout this paper, we use standard vector and tensor notations. In particular, Greek indices take the values 1,2 and the usual summation convention over a subscript and a superscript is employed.

A general theory of a Cosserat surface by Green, Naghdi and Wainwright [8] is developed within the framework of thermodynamics; the derivation in [8] is carried out mainly from an appropriate (two-dimensional) energy equation, together with invariance requirements under superposed rigid body motions. Here we adopt the mode of derivation of the basic theory given by Naghdi [9] or the more generalized form of the theory employed by Green and Naghdi [6]. Special cases of the general theory can be obtained by introduction of suitable constraints, thereby resulting in <u>constrained theories</u>. Alternatively, corresponding special cases can be developed in which the kinematic and the kinetic variables are suitably restricted a priori and then <u>restricted theories</u> are constructed by direct approach. Such special cases of the general theory have been discussed previously by Naghdi [9, Secs. 10 and 15] and by Green and Naghdi [10] and are of particular interest in the context of elastic shell theory. In the present paper, however, we construct another type of restricted theory which is particularly suited for application to problems of fluid sheets. The resulting equations can also be obtained as a constrained case of those given for directed fluid sheets in [6], but we find it more convenient to restrict the kinematic and the kinetic variables at the outset and construct a corresponding restricted theory from an appropriate set of conservation laws in integral form. We therefore confine attention here to a theory in which the director $\underset{\sim}{d}$, while deforming along its length, always remains parallel to a fixed direction specified by a constant unit vector $\mathbf{b}$. Thus, recalling also (9), we have

$$\mathbf{d} = \phi(\theta^{\alpha},t)\mathbf{b} \quad , \quad \mathbf{w} = w(\theta^{\alpha},t)\mathbf{b} \quad , \quad w = \dot{\phi} \quad . \tag{10}$$

Let $\mathcal{P}$, bounded by a closed curve $\partial\mathcal{P}$, be a part of $\mathscr{S}$ occupied by an arbitrary material region of the surface of $\mathcal{C}$ in the present configuration at time t and let

$$\boldsymbol{\nu} = \nu^{\alpha}\mathbf{a}_{\alpha} = \nu_{\alpha}\mathbf{a}^{\alpha} \tag{11}$$

be the outward unit normal to $\partial\mathcal{P}$. It is convenient at this point to define certain additional quantities as follows: The mass density $\rho = \rho(\theta^{\gamma},t)$ of the surface $\mathscr{S}$ in the present configuration; the contact force $\mathbf{N} = \mathbf{N}(\theta^{\gamma},t\,;\boldsymbol{\nu})$ and the contact director force† $\mathbf{M} = \mathbf{M}(\theta^{\gamma},t\,;\boldsymbol{\nu})$, each per unit length of a curve in the present configuration; the assigned force $\mathbf{f} = \mathbf{f}(\theta^{\gamma},t)$ and the assigned director force $\boldsymbol{\ell} = \boldsymbol{\ell}(\theta^{\gamma},t)$, each per unit mass of the surface $\mathscr{S}$; the intrinsic (surface) director force $\mathbf{m}$ per unit area of $\mathscr{S}$; the inertia coefficient $k = k(\theta^{\gamma})$ which is independent of time and is associated with the director velocity; the specific internal energy $\varepsilon = \varepsilon(\theta^{\gamma},t)$; the heat flux $h = h(\theta^{\gamma},t\,;\boldsymbol{\nu})$ per unit time and per unit length of a curve $\partial\mathcal{P}$; the specific heat supply $r = r(\theta^{\gamma},t)$ per unit time; and the element of area $d\sigma$, and the line element ds of the surface $\mathscr{S}$. Further, in view of the assumed form $(10)_1$ for the director, we express $\mathbf{M}$, $\mathbf{m}$ and $\boldsymbol{\ell}$ in terms of their components along and perpendicular to the unit vector $\mathbf{b}$, i.e.,

†The terminology of director couple is also used for $\mathbf{M}$ depending on the physical dimension assumed for the director $\mathbf{d}$. Here we choose $\mathbf{d}$ to have the physical dimension of length so that $\mathbf{M}$ has the same physical dimension as $\mathbf{N}$. For further discussion see [6].

$$\underset{\sim}{M} = M(\theta^\gamma, t\,;\,\underset{\sim}{\nu})\underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{S}(\theta^\gamma, t\,;\,\underset{\sim}{\nu}) \quad , \quad \underset{\sim}{S} \cdot \underset{\sim}{b} = 0 \quad ,$$

$$\underset{\sim}{m} = m(\theta^\gamma, t)\underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{s}(\theta^\gamma, t) \quad , \quad \underset{\sim}{s} \cdot \underset{\sim}{b} = 0 \quad , \tag{12}$$

$$\underset{\sim}{\ell} = \ell(\theta^\gamma, t)\underset{\sim}{b} + \underset{\sim}{b} \times \underset{\sim}{c}(\theta^\gamma, t) \quad , \quad \underset{\sim}{c} \cdot \underset{\sim}{b} = 0 \quad ,$$

where M, m and ℓ are scalar functions and $\underset{\sim}{S}$, $\underset{\sim}{s}$, $\underset{\sim}{c}$ are vector functions of their arguments. Also, it is convenient to decompose the assigned fields $\underset{\sim}{f}$ and ℓ into two parts, one of which represents the three-dimensional body force acting on the continuum which are assumed to be derivable from a potential function $\Omega(\underset{\sim}{r}, \phi)$ and the other which represents the effect of applied surface loads on the major surfaces of the fluid sheet. Thus, we write

$$\underset{\sim}{f} = -\frac{\partial \Omega}{\partial \underset{\sim}{r}} + \overline{\underset{\sim}{f}} \quad , \quad \ell = \left(-\frac{\partial \Omega}{\partial \phi} + \overline{\ell}\right) \quad . \tag{13}$$

With the foregoing definitions of the various field quantities and with reference to the present configuration, the conservation laws for a restricted theory of a Cosserat surface (different from that discussed in [9] or [10]) are:

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \, d\sigma = 0 \quad , \tag{14a}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \underset{\sim}{v} \, d\sigma = \int_{\mathcal{P}} \rho \underset{\sim}{f} \, d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{N} \, ds \quad , \tag{14b}$$

$$\underset{\sim}{b}\frac{d}{dt}\int_{\mathcal{P}} \rho k w \, d\sigma = \underset{\sim}{b}\left[\int_{\mathcal{P}} (\rho \ell - m) d\sigma + \int_{\partial\mathcal{P}} M \, ds\right] + \underset{\sim}{b} \times \left[\int_{\mathcal{P}} (\rho \underset{\sim}{c} - \underset{\sim}{s}) d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{S} \, ds\right] \quad , \tag{14c}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \underset{\sim}{r} \times \underset{\sim}{v} \, d\sigma = \int_{\mathcal{P}} \rho[\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d} \times (\underset{\sim}{b} \times \underset{\sim}{c})] d\sigma + \int_{\partial\mathcal{P}} [\underset{\sim}{r} \times \underset{\sim}{N} + \underset{\sim}{d} \times (\underset{\sim}{b} \times \underset{\sim}{S})] ds \quad , \tag{14d}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho[\epsilon + \Omega + \tfrac{1}{2}(\underset{\sim}{v} \cdot \underset{\sim}{v} + k w^2)] d\sigma = \int_{\mathcal{P}} \rho(r + \overline{\underset{\sim}{f}} \cdot \underset{\sim}{v} + \overline{\ell} w) d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{N} \cdot \underset{\sim}{v} + M w - h) ds \quad . \tag{14e}$$

In the above equations (14a) is a statement of conservation of mass, (14b) the conservation of linear momentum, (14c) that of the conservation of the director momentum, (14d) the conservation of moment of momentum, and (14e) represents the conservation of energy. It should be noted that the quantities M and ℓ

make no contributions to the moment of momentum equation, and the quantities $\underset{\sim}{c}$ and $\underset{\sim}{S}$ make no contribution to the energy equation in the present restricted theory.

Under suitable continuity assumptions, the curve force $\underset{\sim}{N}$, the director force $\underset{\sim}{M}$ and the heat flux h can be expressed as

$$\underset{\sim}{N} = \underset{\sim}{N}^{\alpha}\nu_{\alpha} \quad ,$$

$$M = M^{\alpha}\nu_{\alpha} \quad , \quad \underset{\sim}{S} = \underset{\sim}{S}^{\alpha}\nu_{\alpha} \quad , \quad \underset{\sim}{S}^{\alpha}\cdot\underset{\sim}{b} = 0 \quad , \tag{15}$$

$$h = q^{\alpha}\nu_{\alpha} \quad ,$$

where the fields $\underset{\sim}{N}^{\alpha}, \underset{\sim}{S}^{\alpha}, M^{\alpha}, q^{\alpha}$ are functions of θ^{γ}, t. The five conservation equations in (14) then yield the local equations

$$\rho a^{\frac{1}{2}} = \gamma(\theta^{\gamma}) \quad , \tag{16}$$

$$(a^{\frac{1}{2}}\underset{\sim}{N}^{\alpha})_{,\alpha} + \gamma\underset{\sim}{f} = \gamma\dot{\underset{\sim}{v}} \quad , \tag{17}$$

$$(a^{\frac{1}{2}}M^{\alpha})_{,\alpha} + \gamma\ell = ma^{\frac{1}{2}} + \gamma k\dot{w} \quad , \quad (a^{\frac{1}{2}}\underset{\sim}{S}^{\alpha})_{,\alpha} + \gamma\underset{\sim}{c} = a^{\frac{1}{2}}\underset{\sim}{s} \quad , \tag{18}$$

$$\underset{\sim}{a}_{\alpha} \times \underset{\sim}{N}^{\alpha} + \underset{\sim}{d} \times (\underset{\sim}{b} \times \underset{\sim}{s}) + \underset{\sim}{d}_{,\alpha} \times (\underset{\sim}{b} \times \underset{\sim}{S}^{\alpha}) = \underset{\sim}{0} \quad , \tag{19}$$

$$\rho r - q^{\alpha}{}_{|\alpha} - \rho\dot{\varepsilon} + \underset{\sim}{N}^{\alpha}\cdot\underset{\sim}{v}_{,\alpha} + mw + M^{\alpha}w_{,\alpha} = 0 \quad , \tag{20}$$

where a comma denotes partial differentiation with respect to the surface coordinates θ^{γ} and a vertical line stands for covariant differentiation with respect to the metric tensor of the surface $\mathscr{S}$. It should be noted that the vector fields $\underset{\sim}{S}^{\alpha}$ and $\underset{\sim}{s}$ are workless and do not contribute to the energy equation (20).

3. Incompressible Inviscid Fluids.

Within the scope of the theory of a Cosserat surface, previously in [6] constitutive equations have been obtained for a fluid sheet which model the properties of the three-dimensional isothermal inviscid fluid. For the present restricted theory, we first note that the conditions of incompressibility reduce to the single condition§

$$\frac{d}{dt}\,[\underset{\sim}{a}_1\underset{\sim}{a}_2\underset{\sim}{d}] = 0 \quad . \tag{21}$$

With the help of (9) and (10), the condition (21) can be rewritten as

$$\phi\epsilon^{\alpha\beta}\underset{\sim}{a}_\beta \times \underset{\sim}{b} \cdot \underset{\sim}{v}_{,\alpha} + w\underset{\sim}{a}_3 \cdot \underset{\sim}{b} = 0 \quad , \tag{22}$$

where $\epsilon^{\alpha\beta}$ is the alternating tensor. For an incompressible inviscid fluid at constant temperature, the response functions $\underset{\sim}{N}^\alpha, m, M^\alpha$ are workless, i.e.,

$$\underset{\sim}{N}^\alpha \cdot \underset{\sim}{v}_{,\alpha} + mw + M^\alpha w_{,\alpha} = 0 \quad , \tag{23}$$

provided $\underset{\sim}{v}_{,\alpha}, w$ satisfy the constraint condition (22). It can then be shown that†

$$\underset{\sim}{N}^\alpha = -p_o\phi\epsilon^{\alpha\beta}\underset{\sim}{a}_\beta \times \underset{\sim}{b} \quad , \quad m = -p_o\underset{\sim}{a}_3 \cdot \underset{\sim}{b} \quad , \quad M^\alpha = 0 \quad , \tag{24}$$

where p_o is an arbitrary scalar function of θ^γ, t. Moreover, in a manner similar to that indicated in the appendix of [6], for an inviscid fluid at constant temperature we also have

$$q^\alpha = 0 \quad , \quad \dot{\epsilon} = 0 \quad . \tag{25}$$

It then follows from (20) that

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§In general, there are two conditions of incompressibility in the theory of incompressible directed fluid sheets given in [6]. In our present development, since $\underset{\sim}{d}$ is specified by $(10)_1$, the second condition is satisfied identically and the corresponding pressure (arising from the constraint response) is a part of the response functions for $\underset{\sim}{S}^\alpha$ and $\underset{\sim}{s}$.

†The development leading to (24) is similar to a more general discussion in section 4 of [6].

$$r = 0 \quad . \tag{26}$$

Also, for the present restricted theory the moment of momentum equation (19) is satisfied if

$$\mathbf{S}^{\alpha}\phi_{,\alpha} + \mathbf{s}\phi + p_o \phi \epsilon^{\alpha\beta} \mathbf{a}_{\beta} (\mathbf{b} \cdot \mathbf{a}_{\alpha}) = \mathbf{0} \quad . \tag{27}$$

The last equation is consistent with the restrictions $\mathbf{S}^{\alpha} \cdot \mathbf{b} = 0$, $\mathbf{s} \cdot \mathbf{b} = 0$ given in (15) and (12). From (18) and (27), after eliminating $\mathbf{s}$, we obtain the following equation for $\mathbf{S}^{\alpha}$:

$$(\phi a^{\frac{1}{2}} \mathbf{S}^{\alpha})_{,\alpha} + p_o \phi a^{\frac{1}{2}} \epsilon^{\alpha\beta} \mathbf{a}_{\beta} (\mathbf{b} \cdot \mathbf{a}_{\alpha}) + \gamma \phi \mathbf{c} = \mathbf{0} \quad . \tag{28}$$

4. Water Waves for Nonhomogeneous Stream of Variable Depth.

Our previous discussion of water waves in [6] was confined to one-dimensional flows. Here we consider two-dimensional flows and at the same time allow our model to reflect the (three-dimensional) properties of a nonhomogeneous incompressible fluid. Let $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ be a set of right-handed constant orthonormal base vectors associated with rectangular Cartesian axes and let the position vector $\underset{\sim}{r}$ in $(7)_1$ and the director $\underset{\sim}{d}$ in $(10)_1$ be represented as

$$\underset{\sim}{r} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \psi\underset{\sim}{e}_3 \quad , \quad \underset{\sim}{d} = \phi\underset{\sim}{e}_3 \quad , \quad \underset{\sim}{b} = \underset{\sim}{e}_3 \quad , \tag{29}$$

where x, y, ψ, ϕ are functions of θ^1, θ^2, t. The velocity $\underset{\sim}{v}$ and the director velocity now take the forms

$$\underset{\sim}{v} = u\underset{\sim}{e}_1 + v\underset{\sim}{e}_2 + \lambda\underset{\sim}{e}_3 \quad , \quad \underset{\sim}{w} = w\underset{\sim}{e}_3 \quad , \tag{30}$$

where

$$u = \dot{x} \quad , \quad v = \dot{y} \quad , \quad \lambda = \dot{\psi} \quad , \quad w = \dot{\phi} \tag{31}$$

and we note that the velocity components u, v, λ, w may be regarded as functions of either θ^1, θ^2, t or of x, y, t. From (30) follow the expressions

$$\dot{\underset{\sim}{v}} = \dot{u}\underset{\sim}{e}_1 + \dot{v}\underset{\sim}{e}_2 + \dot{\lambda}\underset{\sim}{e}_3 \quad , \quad \dot{\underset{\sim}{w}} = \dot{w}\underset{\sim}{e}_3 \tag{32}$$

and

$$\begin{aligned} \dot{u} &= u_t + uu_x + vu_y \quad , \quad \dot{v} = v_t + uv_x + vv_y \quad , \\ \dot{\lambda} &= \lambda_t + u\lambda_x + v\lambda_y \quad , \quad \dot{w} = w_t + uw_x + vw_y \quad , \end{aligned} \tag{33}$$

where as in section 1 the subscripts x, y, t designate partial differentiation with respect to x, y, t when u, v, λ, w are regarded as functions of x, y, t. Also, by (7), (8) and (29), the base vectors $\underset{\sim}{a}_\alpha$ and the unit normal $\underset{\sim}{a}_3$ can be expressed in the forms

$$
\begin{aligned}
\underset{\sim}{a}_1 &= \frac{\partial x}{\partial \theta^1}\underset{\sim}{e}_1 + \frac{\partial y}{\partial \theta^1}\underset{\sim}{e}_2 + \frac{\partial \psi}{\partial \theta^1}\underset{\sim}{e}_3 \quad , \\
\underset{\sim}{a}_2 &= \frac{\partial x}{\partial \theta^2}\underset{\sim}{e}_1 + \frac{\partial y}{\partial \theta^2}\underset{\sim}{e}_2 + \frac{\partial \psi}{\partial \theta^2}\underset{\sim}{e}_3 \quad , \\
\underset{\sim}{a}_3 a^{\frac{1}{2}} &= (-\psi_x\underset{\sim}{e}_1 - \psi_y\underset{\sim}{e}_2 + \underset{\sim}{e}_3)\,\frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad .
\end{aligned}
\tag{34}
$$

With the use of (30) and (33), the incompressibility condition (22) assumes the simpler form

$$\phi(u_x + v_y) + w = 0 \tag{35}$$

and, with the help of (24), (32) and (33), the equations of motion (17) and $(18)_1$ reduce to the forms

$$\gamma\dot{u} = \gamma\underset{\sim}{f}\cdot\underset{\sim}{e}_1 - \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)}\,p_x \quad , \tag{36}$$

$$\gamma\dot{v} = \gamma\underset{\sim}{f}\cdot\underset{\sim}{e}_2 - \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)}\,p_y \quad , \tag{37}$$

$$\gamma\dot{\lambda} = \gamma\underset{\sim}{f}\cdot\underset{\sim}{e}_3 \quad , \tag{38}$$

$$\gamma k\dot{w} = \gamma\ell + \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)}\,\frac{p}{\phi} \quad , \tag{39}$$

where p is given by

$$p = p_o\phi \quad . \tag{40}$$

It remains to specify values for the coefficients γ, k, the assigned force $\underset{\sim}{f}$ and the assigned director force $\underset{\sim}{\ell}$. For this purpose, we are guided by the corresponding fluid sheet in the three-dimensional theory in which an incompressible homogeneous fluid under gravity‡ $-g^*\underset{\sim}{e}_3$ flows over a bed specified by the position vector

‡We use g^* (instead of g) for gravity and reserve the symbol g for later use in Eq. (48).

$$\underset{\sim}{p} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \alpha(x,y)\underset{\sim}{e}_3 \tag{41}$$

and we specify the surface of the fluid by

$$\underset{\sim}{p} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \beta(x,y,t)\underset{\sim}{e}_3 \quad . \tag{42}$$

In (41), α is a given function of x,y but β in (42) depends on x,y,t. At the surface (42) of the stream there is a constant pressure p_o and a constant normal surface tension T. At the bed the (unknown) pressure $\overline{p}$ depends on x,y and t. Thus, the fluid moves with the surface (42) and the normal pressure p^* at this surface is

$$p^* = p_o - q \quad , \tag{43}$$

where

$$q = \frac{T\{(1+\beta_y^2)\beta_{xx} - 2\beta_x\beta_y\beta_{xy} + (1+\beta_x^2)\beta_{yy}\}}{(1+\beta_x^2+\beta_y^2)^{3/2}} \quad . \tag{44}$$

At the bed (41) the normal velocity of the fluid is zero and the pressure p^* takes the value

$$p^* = \overline{p}(x,y,t) \quad , \tag{45}$$

where $\overline{p}$ is to be determined.

To proceed further, we introduce a set of convected coordinates θ^i $(i = 1,2,3)$, let the surface $\theta^3 = 0$ coincide with the surface $\mathscr{S}$, and consider the three-dimensional region of space between the surfaces (41) and (42) occupied by the fluid. Any point in this three-dimensional region is then specified by

$$\underset{\sim}{p} = x_1\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + (\psi + \theta^3\phi)\underset{\sim}{e}_3 \quad . \tag{46}$$

We also suppose that the surfaces $\theta^3 = \omega - \frac{1}{2}$ and $\theta^3 = \omega + \frac{1}{2}$, with ω a constant to be determined later, are coincident with (41) and (42), respectively, so that

$$\alpha = \psi + (\omega - \tfrac{1}{2})\phi \quad , \quad \beta = \psi + (\omega + \tfrac{1}{2})\phi \quad . \tag{47}$$

The mass density ρ^* of the fluid is assumed to vary with depth so that $\rho^* = \rho^*(\theta^3)$.

The base vectors $\underset{\sim}{g}_i$ in the above-mentioned three-dimensional region, as well as their conjugates $\underset{\sim}{g}^i$ and related results, can be calculated from (46) but we only record here the formulas

$$\begin{aligned} g^{\frac{1}{2}} &= [\underset{\sim}{g}_1 \underset{\sim}{g}_2 \underset{\sim}{g}_3] = \phi \, \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \\ g^{\frac{1}{2}} \underset{\sim}{g}^3 &= [-(\psi_x + \theta^3\phi_x)\underset{\sim}{e}_1 - (\psi_y + \theta^3\phi_y)\underset{\sim}{e}_2 + \underset{\sim}{e}_3] \, \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad . \end{aligned} \tag{48}$$

We now make use of the results derived in [6] in order to obtain explicit values of $\gamma, \underset{\sim}{f}, \ell$ and the parameter ω. First, in relation to the top and bottom surfaces of the fluid, *we choose the surface $\theta^3 = 0$ so that the center of mass of the* (three-dimensional) fluid region under consideration always lies on this surface and we then identify this surface with the surface $\mathscr{S}$ in the theory of Cosserat surface. This leads us to impose the condition

$$\int_{\omega - \frac{1}{2}}^{\omega + \frac{1}{2}} \rho^*(\theta^3)\theta^3 d\theta^3 = 0 \qquad (-\tfrac{1}{2} < \omega < \tfrac{1}{2}) \quad , \tag{49}$$

which determines ω. Then, recalling the results in [6], we have†

$$\rho a^{\frac{1}{2}} = \gamma = \int_{\omega - \frac{1}{2}}^{\omega + \frac{1}{2}} \rho^* g^{\frac{1}{2}} d\theta^3 \quad , \quad \gamma k = \int_{\omega - \frac{1}{2}}^{\omega + \frac{1}{2}} (\theta^3)^2 \rho^* g^{\frac{1}{2}} d\theta^3 \tag{50}$$

or equivalently

$$\gamma = \kappa\phi \, \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \quad \gamma k = \kappa_2 \phi \, \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \tag{51}$$

where

$$\kappa = \int_{\omega - \frac{1}{2}}^{\omega + \frac{1}{2}} \rho^* d\theta^3 \quad , \quad \kappa_2 = \int_{\omega - \frac{1}{2}}^{\omega + \frac{1}{2}} \rho^*(\theta^3)^2 d\theta^3 \quad . \tag{52}$$

---

†See particularly equations (3.7) and (3.15) in [6].

Also, with the use of (13), (51) and (48), from Eqs. (3.15) of [6] for the assigned force and the assigned director force we obtain

$$
\begin{aligned}
\gamma \underset{\sim}{f} &= [\{(p_o-q)\beta_x - \bar{p}\alpha_x\}\underset{\sim}{e}_1 + \{(p_o-q)\beta_y - \bar{p}\alpha_y\}\underset{\sim}{e}_2 \\
&\quad + (q - p_o + \bar{p} - g^* \kappa\phi)\underset{\sim}{e}_3]\ \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \\
\gamma \ell &= [(q-p_o)(\tfrac{1}{2}+\omega) + \bar{p}(-\tfrac{1}{2}+\omega)]\ \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \\
\gamma \underset{\sim}{c} &= [(p_o-q)(\tfrac{1}{2}+\omega)\beta_x - \bar{p}(-\tfrac{1}{2}+\omega)\alpha_x]\underset{\sim}{e}_1\ \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \\
&\quad - [(p_o-q)(\tfrac{1}{2}+\omega)\beta_y - \bar{p}(-\tfrac{1}{2}+\omega)\alpha_y]\underset{\sim}{e}_2\ \frac{\partial(x,y)}{\partial(\theta^1,\theta^2)} \quad , \\
\Omega &= g^* \psi \quad .
\end{aligned}
\tag{53}
$$

Substitution of (51) to (53) into (36) to (39) results in the differential equations of motion

$$
\begin{aligned}
\kappa\phi\dot{u} &= -p_x + (p_o-q)\beta_x - \bar{p}\alpha_x \quad , \\
\kappa\phi\dot{v} &= -p_y + (p_o-q)\beta_y - \bar{p}\alpha_y \quad , \\
\kappa\phi\dot{\lambda} &= q - p_o + \bar{p} - g^*\kappa\phi \quad , \\
\kappa_2\phi\dot{w} &= (q-p_o)(\tfrac{1}{2}+\omega) + \bar{p}(-\tfrac{1}{2}+\omega) + \frac{p}{\phi} \quad .
\end{aligned}
\tag{54}
$$

Moreover, since the bed of the stream is stationary, from (47) and $(33)_{3,4}$ we have

$$
\dot{\alpha} = u\alpha_x + v\alpha_y = \dot{\psi} + (\omega - \tfrac{1}{2})\dot{\phi} = \lambda + (\omega - \tfrac{1}{2})w \quad . \tag{55}
$$

The above system of equations is independent of the remaining equations (27) and (28) which involve $\underset{\sim}{S}^{\alpha}, \underset{\sim}{s}$. The fields $\underset{\sim}{S}^{\alpha}, \underset{\sim}{s}$ correspond to appropriate constraint responses for the restricted motion (10) considered in the present paper; and, since they are not completely determined by (27) and (28), there is some arbitrariness in their specification. We do not need to consider equations (27) and (28) further here.

To summarize the results obtained so far, we observe that the propagation of water waves in a nonhomogeneous incompressible inviscid fluid is governed by the system of differential equations (35), (54) and (55) with the coefficients κ, κ_2 given by (52) and ω determined from (49). When the fluid is homogeneous with ρ^* a constant, it readily follows from (49) and (52) that

$$\omega = 0 \quad , \quad \kappa = \rho^* \quad , \quad \kappa_2 = \frac{\rho^*}{12} \quad . \tag{56}$$

With values (56), the differential equations (35), (54) and (55) become identical with those derived previously [7] from the three-dimensional theory of homogeneous incompressible inviscid fluids. For unidirectional flow along the x-direction, the equations reduce to those obtained in [6] by a direct approach.

The above equations have been derived from the integral balance equations (14a-e). With the help of (24), (25), (26), (30), (51) and (53), these integral balance laws can be reduced to the forms:

$$\frac{d}{dt} \int_{\mathcal{P}} \kappa \phi \, dxdy = 0 \quad , \tag{57}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \kappa \phi u \, dxdy = \int_{\mathcal{P}} [(p_o - q)\beta_x - \bar{p}\alpha_x] dxdy - \int_{\partial\mathcal{P}} p \, dy \quad , \tag{58}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \kappa \phi v \, dxdy = \int_{\mathcal{P}} [(p_o - q)\beta_y - \bar{p}\alpha_y] dxdy + \int_{\partial\mathcal{P}} p \, dx \quad , \tag{59}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \kappa \phi \lambda \, dxdy = \int_{\mathcal{P}} (q - p_o + \bar{p} - g^* \kappa \phi) dxdy \quad , \tag{60}$$

$$\frac{d}{dt} \int_{\mathcal{P}} \kappa_2 \phi w \, dxdy = \int_{\mathcal{P}} [\frac{p}{\phi} + (q - p_o)(\tfrac{1}{2} + \omega) - \bar{p}(\tfrac{1}{2} - \omega)] dxdy \quad , \tag{61}$$

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{P}} \tfrac{1}{2} \phi [\kappa(u^2 + v^2 + \lambda^2) + \kappa_2 w^2 + 2g^* \kappa \psi] dxdy \\ = \int_{\mathcal{P}} \{(p_o - q)[\beta_x u + \beta_y v - \lambda - (\tfrac{1}{2} + \omega)w] - \bar{p}[\alpha_x u + \alpha_y v - \lambda + (\tfrac{1}{2} - \omega)w]\} \, dxdy \\ - \int_{\partial\mathcal{P}} p(u\,dy - v\,dx) \quad . \end{aligned} \tag{62}$$

5. Solitary Waves on a Stream with Level Bed.

When the bed of the stream is level and the fluid is homogeneous with a constant value for ρ^*, it has been shown in [5] that the system of equations (35), (54) and (55) admits a solution in the form of a solitary wave. A similar solution is possible here even when the fluid is nonhomogeneous with the mass density ρ^* varying with depth. Omitting details, for a wave travelling along the x-axis with a speed c, the solitary wave has the form

$$\frac{\phi}{h} = 1 + \mu^2 \operatorname{sech}^2 [(\frac{\kappa}{(1-2\omega)^2 \kappa + 4\kappa_2})^{\frac{1}{2}} \frac{c_o \mu (x-ct)}{ch}] \quad , \tag{63}$$

where‡

$$\mu = \frac{(c^2 - c_o^2)^{\frac{1}{2}}}{c_o} \quad , \quad c_o^2 = gh(1-2\omega) \quad , \quad c^2 > c_o^2 \quad . \tag{64}$$

‡Previously [5] the notation λ was employed for a quantity corresponding to μ defined by $(64)_1$. In the present paper, the symbol λ is utilized for a different purpose in (35).

6. Hydraulic Jumps.

In a previous paper [6, Sec. 8] we discussed hydraulic jumps at a step in the horizontal bed of a stream. Here we extend the discussion to nonhomogeneous fluids in which the mass density varies with depth, and at the same time indicate the nature of an additional class of possible solutions which was not noted in [6].

We assume steady flow of the fluid parallel to the x-direction in a stream whose bottom is level except for a finite jump at the origin, and thus we specify α by

$$\left.\begin{aligned} &\psi + (\omega - \tfrac{1}{2})\phi = \alpha = \begin{cases} 0 \text{ for } x < 0 \ , \\ d \text{ for } x > 0 \ , \end{cases} \\ &\psi_x = (\tfrac{1}{2} - \omega)\phi_x \text{ for } x \lessgtr 0 \ , \end{aligned}\right\} \tag{65}$$

where d is a constant. We neglect surface tension so that $q = 0$, for convenience set the surface pressure $p_0 = 0$, and consider only steady motions on either side of $x = 0$ with the possibility that the wave height ϕ at $x = 0$ has a stationary finite discontinuity. At $x = 0$, we assume that ϕ, u, λ, w, p change from the values $\phi_1, u_1, \lambda_1, w_1, p_1$ to the values $\phi_2, u_2, \lambda_2, w_2, p_2$. We may than utilize the one-dimensional form of the equations (57), (58), (60), (61) and (62) with $v = 0$ in a usual way to obtain the appropriate jump conditions. Thus[†]

$$u_2\phi_2 = u_1\phi_1 = n \ , \tag{66}$$

$$\kappa n(u_2 - u_1) = p_1 - p_2 + X \ , \tag{67}$$

$$\lambda_1 = \lambda_2 = (\tfrac{1}{2} - \omega)w_1 = (\tfrac{1}{2} - \omega)w_2 \ , \tag{68}$$

$$\tfrac{1}{2}\kappa n(u_2^2 - u_1^2) + (\tfrac{1}{2} - \omega)g^* n\kappa(\phi_2 - \phi_1) + g^* n\kappa d = p_1u_1 - p_2u_2 \ . \tag{69}$$

[†]The constant n in (66) to (69) corresponds to k in the corresponding jump conditions given in [6, Sec. 8].

In [6, Sec. 8] the discussion was limited to the special case in which $X = 0$. However, in general, the quantity X could be nonzero; it arises from the integrated value $\bar{p}\alpha_x$ in (58) since $\bar{p}\alpha_x$ may become large at the step, $\bar{p}$ being the pressure on the bottom of the stream. The term X on the right-hand side of (67), therefore, represents the resultant force exerted by the step on the fluid measured along the positive x-direction.[§] Assuming that the fluid remains in contact with the step throughout the motion, we impose the conditions

$$
\begin{aligned}
X &\leqq 0 \text{ when } d > 0 \quad , \\
X &\geqq 0 \text{ when } d < 0 \quad .
\end{aligned}
\tag{70}
$$

In order to illustrate the differences which arise when $X \neq 0$, we consider now the simplest problem discussed in [6, Sec. 8]. Thus, when the stream is level on either side of the step, from (54) we have

$$
\begin{aligned}
\phi_1 &= h \quad , \quad p_1 = (\tfrac{1}{2} - \omega) g^* \kappa h^2 \quad , \quad u_1 = \frac{n}{h} \quad , \\
\phi_2 &= H \quad , \quad p_2 = (\tfrac{1}{2} - \omega) g^* \kappa H^2 \quad , \quad u_2 = \frac{n}{H} \quad .
\end{aligned}
\tag{71}
$$

For convenience, we introduce the notations

$$
L = \frac{H}{h} \quad , \quad \tau = \frac{n^2}{g^*(1-2\omega)h^3}
\tag{72}
$$

and then substitute (71) into (67) and (69) and also use (66) to obtain

$$
\begin{aligned}
\frac{d}{h} &= \frac{(1-2\omega)(L-1)}{L^2} [\tfrac{1}{2}\tau(L+1) - L^2] \quad , \\
-\frac{2X}{\kappa g^* h^2} &= \frac{(1-2\omega)(L-1)}{L} [2\tau - L - L^2] \quad .
\end{aligned}
\tag{73}
$$

---

§ Similar resultant forces occur in jump conditions utilized by Caulk [11] in his treatment of the problem of fluid flow under a sluice gate based on the two-dimensional theory of directed fluid sheets [5,6].

To simplify the discussion that follows, we set

$$4\tau_1 = \tau + (\tau^2+8\tau)^{\frac{1}{2}} \quad , \quad 4\tau_2 = -\tau + (\tau^2+8\tau)^{\frac{1}{2}} \quad ,$$
$$\tau_3 + \tfrac{1}{2} = (\tfrac{1}{4} + 2\tau)^{\frac{1}{2}} \tag{74}$$

and then rewrite $(73)_{1,2}$ as

$$\frac{d}{h} = \frac{(1-2\omega)(L-1)}{L^2}(\tau_1 - L)(L+\tau_2) \quad ,$$
$$-\frac{2X}{\kappa g^* h^2} = \frac{L-1}{L}(1-2\omega)(\tau_3 - L)(L+\tau_3+1) \quad . \tag{75}$$

If we confine attention to values of $\tau > 1$, then $\tau_1 > 1$ and $\tau_3 > 1$ for all values of ω.

In the rest of this discussion, we confine attention to homogeneous fluids in which

$$\omega = 0 \quad , \quad \kappa = \rho^* \quad , \quad \tau_3 + \tfrac{1}{2} = (\tfrac{1}{4} + 2\tau)^{\frac{1}{2}} \quad (\tau > 1, \ \tau_3 > 1) \quad , \tag{76}$$

but a parallel discussion can be carried out for nonhomogeneous fluids with $\omega \neq 0$. The previous solution [6, Sec. 8] corresponds to the case for which

$$L = \tau_3 \quad , \quad X = 0 \quad , \quad \frac{d}{h} = \frac{(\tau_3 - 1)^3}{4\tau_3} \quad . \tag{77}$$

On the other hand, if we take the value of d to be that specified by $(77)_3$, then in addition to $(77)_1$ we find a second possible solution for L, namely

$$L = \frac{\tau_3+1}{8\tau_3} + \{(\frac{\tau_3+1}{8\tau_3})^2 + \frac{\tau_3+1}{4}\}^{\frac{1}{2}} \quad , \tag{78}$$

which satisfies $(75)_1$. It can be verified from $(75)_2$ that the second value of L given by (78) corresponds to

$$-X > 0 \quad , \quad d > 0 \tag{79}$$

in comformity with (70).

To continue, we observe that for values of d/h specified by

$$0 < \frac{d}{h} < \frac{(\tau_3 - 1)^3}{4\tau_3} \tag{80}$$

there are in general two positive values of L which satisfy $(75)_1$ but only one of these is compatible with (79). Further, if

$$\frac{(\tau_3 - 1)^3}{4\tau_3} \leq \frac{d}{h} < \frac{d_1}{h} \quad , \tag{81}$$

where d_1 is a maximum positive value of d for varying L in $(75)_1$, then there are two values of L corresponding to each value of d/h and both of these satisfy (79).

A more general discussion of the other problems discussed in [6, Sec. 8] can be given in a similar manner, but we do not pursue the matter further.

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After a brief discussion of some undesirable features of a number of different partial differential equations often employed in the existing literature on water waves, a relatively simple restricted theory is constructed by a direct approach which is particularly suited for applications to problems of fluid sheets. The rest of the paper is concerned with a derivation of a system of nonlinear differential equations (which may include the effects of gravity and surface tension) governing the two-dimensional motion of incompressible next pag

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inviscid fluids for propagation of fairly long waves in a nonhomogeneous stream of water of variable initial depth, as well as some new results pertaining to hydraulic jumps. The latter includes an additional class of possible solutions not noted previously.